1 Electromagnetic waves

1.1 Wave equation

The wave equation of electromagnetic fields follows directly from Maxwell's equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

$\vec{\nabla} \cdot \vec{D} = \rho$ charges are sources of electric fields

$\vec{\nabla} \cdot \vec{B} = 0$ no magnetic sources

Polarisation and magnetisation are defined by material equations

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

with

$\vec{E}(\vec{r},t)$	electric field
$\vec{H}(\vec{r},t)$	magnetic field
$\vec{D}(\vec{r},t)$	dielectric displacement
$\vec{B}(\vec{r},t)$	magnetic flux density (induction)
$\vec{P}(\vec{r},t)$	electric polarisation
$\vec{M}(\vec{r},t)$	magnetic polarisation (magnetisation)
ε_0	dielectric permeability (of vacuum)
μ_0	magnetic permeability (field constant)
$\vec{j}$	current density (of free charges)

In what follows we consider the electric magnetic field in a typical dielectric, i.e. we have neither free charges nor currents due to free charges ($\rho = 0$, $\vec{j} = 0$), and our material is considered to be non-magnetic ($\vec{M} = 0$ or $\mu = 1$)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \varepsilon_0 \vec{E} \;\Rightarrow\; \vec{P} = \varepsilon_0 (\varepsilon - 1) \vec{E} = \varepsilon_0 \chi \vec{E}$$

If ε is a scalar, then $\vec{P} \parallel \vec{E} \Rightarrow \vec{D} \parallel \vec{E}$ and the medium is called isotropic. Furthermore, if the dielectric constant ε does not depend on space vector $\vec{r}$ (i.e. $\vec{\nabla}\varepsilon = 0$) our medium is homogeneous, and if it does not depend on the amplitude $|\vec{E}|$ of the electric field it behaves linearly.

Thus, for an isotropic, homogeneous linear medium with $\varepsilon\varepsilon_0$ (alternatively the relation $\varepsilon \equiv \varepsilon_r$ is used in text books) we have

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \varepsilon \vec{E} = \varepsilon_0 (1+\chi) E$$

or $$\vec{P} = \varepsilon_0 \chi \vec{E}$$

$$\chi = \varepsilon - 1 \qquad \text{: electric susceptibility}$$

It is clear that – in a microscopic view – atoms or molecules are never isotropic or homogeneous. However, if the number of atoms to be considered in a volume λ^3 is large (here λ is the wavelength of light) and where the atoms are regularly distributed in this volume, the anisotropy averages out and the medium behaves homogeneous.

With these assumptions ($\vec{j}$ = 0, ρ = 0, $\vec{M}$ = 0) we obtain the relations

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E} + \mu_0 \frac{\partial \vec{H}}{\partial t} = 0$$

$$\vec{\nabla} \times \vec{H} - \varepsilon\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$$

↥ valid for $\partial\varepsilon/\partial t = 0$, or more correctly: $\varepsilon(t)$ changes slowly with time compared to $E(t)$

$$\vec{\nabla} \cdot (\varepsilon\varepsilon_0 \vec{E}) = 0$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

The wave equation is obtained by applying the following operation $\vec{\nabla} \times \vec{0} = \vec{0}$ on the two equations above:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) + \mu_0 \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) + \mu_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right) = 0$$

This leads to

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} + \mu_0 \frac{\partial}{\partial t}\left(\varepsilon\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = \vec{\nabla} \times \vec{\nabla} \times \vec{E} + \mu_0 \varepsilon\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

For a homogeneous medium we get

$$\vec{\nabla} \cdot \left(\varepsilon \vec{E}\right) = \underbrace{\vec{\nabla}\varepsilon\varepsilon_0}_{\equiv 0} \vec{E} + \varepsilon\varepsilon_0 \vec{\nabla} \cdot \vec{E} - \varepsilon\varepsilon_0 \vec{\nabla} \cdot \vec{E}$$

$$\Leftrightarrow \quad \vec{\nabla} \cdot \left(\varepsilon\varepsilon_0 \vec{E}\right) = 0 \quad \Leftrightarrow \quad \vec{\nabla} \cdot \vec{E} = 0$$

By using the following vector relation

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}\,\underbrace{(\vec{\nabla} \cdot \vec{E})}_{\equiv 0} - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

we obtain the wave equation for charge-free ($\rho = 0$) and non-magnetic ($\mu = 1$) homogeneous dielectric media

$$\nabla^2 \vec{E} = \mu_0 \,\varepsilon \varepsilon_0 \,\frac{\partial^2 \vec{E}}{\partial t^2}$$

In an analogous way we can calculate the wave equation for the magnetic field in the form

$$\nabla^2 \vec{H} = \mu_0 \,\varepsilon \,\varepsilon_0 \,\frac{\partial^2 \vec{H}}{\partial t^2}$$

1.2 Monochromatic plane waves

Monochromatic plane waves are solutions of the wave equation:

$$\vec{E} = \frac{1}{2}\,\vec{E}_0 \exp(i\omega t - i\vec{k}\cdot\vec{r}) + c.c$$

$$= \vec{E}_0 \cos(\omega t - \vec{k}\cdot\vec{r}) \;,$$

$$\vec{H} = \frac{1}{2}\,\vec{H}_0 \;\exp(i\omega t - i\vec{k}\cdot\vec{r}) + c.c.$$

$$= \vec{H}_0 \cos(\omega t - \vec{k}\cdot\vec{r}) \;,$$

with amplitude vectors $\vec{E}_0, \vec{H}_0$ and wave vector $\vec{k}$. Here $\left|\vec{k}\right| = 2\pi\, n/\lambda$ is the wave number, where n is refractive index and λ is the wavelength in vacuum.

For the wave equation we have used the relation

$$\vec{\nabla}\cdot\vec{D} = 0 \;\Rightarrow\; \vec{\nabla}\cdot(\varepsilon\varepsilon_0\,\vec{E}) = 0 \;\Rightarrow\; \vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(\vec{E}_0 \exp(i\omega\, t - i\vec{k}\cdot\vec{r}\,)\right)$$

$$= \frac{\partial}{\partial x}\left(E_{0,x}\;\exp(i\omega\, t - i[k_x\, x + k_y\, y + k_z\, z])\right) + \frac{\partial}{\partial y}\ldots$$

$$= (-\,i\,E_{0,x}\,k_x - i\,E_{0,y}\,k_y - i\,E_{0,z}\,k_z)\,\exp(i\omega t - i\vec{k}\cdot\vec{r}\,)$$

$$= -i\,\vec{k}\cdot\vec{E} \;.$$

As a result we obtain the relation $\vec{\nabla}\cdot\vec{E} = -i\vec{k}\cdot\vec{E}$ and in analogous way $\vec{\nabla}\cdot\vec{H} = -i\vec{k}\cdot\vec{H}$. From $\vec{k}\cdot\vec{E}_0 = 0$ and $\vec{k}\cdot\vec{H}_0 = 0$ we get $\vec{k}\perp\vec{E}_0$ und $\vec{k}\perp\vec{H}_0$.

Starting from the Maxwell relation

$$\vec{\nabla}\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \ ,$$

we consider the left and right hand side for the case of a plane wave

$$-\mu_0 \frac{\partial \vec{H}}{\partial t} = -\mu_0\, i\omega \vec{H}_0 \exp(i\omega t - i\vec{k}\cdot\vec{r}) \ .$$

and

$$\vec{\nabla}\times\vec{E} = -i\vec{k}\times\vec{E}_0 \exp(i\omega t - i\vec{k}\cdot\vec{r})$$

$$= -i\omega\mu_0 \vec{H}_0 \exp(i\omega t - i\vec{k}\cdot\vec{r}) \ .$$

A comparison yields

$$\vec{H}_0 = \frac{1}{\omega\mu_0}\vec{k}\times\vec{E}_0$$

The three vectors $\vec{E}_0, \vec{H}_0$ und $\vec{k}$ therefore form an orthogonal system

$$\vec{E}_0 \perp \vec{H}_0 \ , \ \vec{E}_0 \perp \vec{k} \ , \ \vec{H}_0 \perp \vec{k} \ .$$

For the wave vector $\vec{k}$ coinciding with the direction of wave propagation, it follows that plane waves in homogeneous, isotropic, non-magnetic media do not have field components in propagation direction. This is valid for both linear (ε = *const*.) and nonlinear ($\varepsilon = \varepsilon(|E|)$) media. Solutions that fulfil these requirements are called TEM waves. However, if the propagation of waves is additionally limited by appropriate boundary conditions, as it is the case for optical waveguides, field components in propagation direction may exist.

There remain two classes of solutions for $\vec{k}\,\|\,\hat{z}$, $\vec{E}\perp\vec{H}$:

(E_x, H_y) : polarisation along $\hat{x}$

(E_y, H_x) : polarisation along $\hat{y}$

Insertion into the wave equation ($\hat{x}$ polarisation) yields

$$\nabla^2\vec{E} \underset{E_y=E_z=0}{\overset{\uparrow}{=}} \nabla^2 E_x = \underbrace{\frac{\partial^2 E_x}{\partial x^2}}_{\equiv 0} + \underbrace{\frac{\partial^2 E_x}{\partial y^2}}_{\equiv 0} + \frac{\partial^2 E_x}{\partial z^2} = \mu_0\varepsilon\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \ .$$

The result is

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \, \varepsilon \, \varepsilon_0 \, \frac{\partial^2 E_x}{\partial t^2} \; ,$$

$$\frac{\partial^2 H_y}{\partial z^2} = \mu_0 \, \varepsilon \, \varepsilon_0 \, \frac{\partial^2 H_y}{\partial t^2} \; .$$

Generalized solutions are of the form

$$E_x^{\pm}(z,t) = \frac{1}{2} E_{0,x}^{\pm} \exp(i\omega\, t \mp i\, k\, z) \; .$$

Let us consider the solution E_x^+: An observer that moves together with this wave always „sees" a constant field amplitude (or a constant phase of the wave), which means that the argument of the exp-function is a constant:

$$\omega t - k\, z = const. = \varphi_0 \; ,$$

Here φ_0 is an arbitrary phase which determines the value of the (constant) field amplitude.

The observer moves in positive z direction with the phase velocity of the wave (i.e. with the speed of light)

$$c = \frac{d\, z}{d\, t} = \frac{\omega}{k} \; .$$

For the solution E_x^- we use the negative sign for the wave vector k – in this case the wave moves in negative z direction.

Calculation of the second derivatives for z and t and insertion into the wave equation results in the dispersion relation of the wave

$$\frac{\partial^2 E_x^{\pm}}{\partial z^2} = -k^2 \, E_x^{\pm} \quad , \quad \frac{\partial^2 E_x^{\pm}}{\partial t^2} = -\omega^2 \, E_x^{\pm} \; ,$$

$$-k^2 \, E_x^{\pm} = -\mu_0 \, \varepsilon \, \varepsilon_0 \, \omega^2 \, E_x^{\pm} \; ,$$

$$\Rightarrow \qquad k^2 = \mu_0 \, \varepsilon \, \varepsilon_0 \, \omega^2$$

or

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \, \varepsilon \, \varepsilon_0}} \quad \Rightarrow \; k = \omega \sqrt{\mu_0 \, \varepsilon \, \varepsilon_0} \; .$$

In vacuum $(\varepsilon = 1)$ the speed of light is defined by

$$c = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}} \approx 3 \times 10^8 \text{ m/s}$$

and in media with $\varepsilon \neq 1$

$$c_n = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}}\,\frac{1}{\sqrt{\varepsilon}} = \frac{c}{n}$$

with refractive index $n = \sqrt{\varepsilon}$. The amplitude of the magnetic field $H_y^{\pm}$ may be obtained by

$$\frac{\partial H_y^{\pm}}{\partial z} = -\varepsilon\,\varepsilon_0\,\frac{\partial E_x^{\pm}}{\partial t} \quad .$$

For the case "+" we have

$$-i k H_y^{+} = -i\omega\,\varepsilon\,\varepsilon_0\,E_x^{+}$$

$$\Rightarrow \quad H_y^{+} = \frac{1}{\eta}\,E_x^{+}$$

with $\qquad \eta = \sqrt{\dfrac{\mu_0}{\varepsilon\,\varepsilon_0}}$

In vacuum η is equal to the impedance η_0 = 377 Ω.

The wave equation is a linear differential equation. Therefore the superposition of individual solutions is again a solution of the wave equation:

$$E_x(z,t) = E_{x,0}^{+}\,\exp(i\omega\,t - i k z) + E_{x,0}^{-}\exp(i\omega\,t + i k z) \quad ,$$

$$H_y(z,t) = H_{y,0}^{+}\,\exp(i\omega\,t - i k z) + H_{y,0}^{-}\exp(i\omega\,t + i k z) \quad .$$

The amplitudes $E_{x,0}^{+}$, $E_{x,0}^{-}$ and $H_{y,0}^{+}$, $H_{y,0}^{-}$, respectively, are determined by boundary conditions.

1.3 Energy flux and Poynting theorem

The aim of this section is to describe the energy density (energy per volume) of an electromagnetic wave in a dielectric medium. Starting point are again Maxwell's equations together with material equations describing the material properties:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \ ,$$

$$\vec{\nabla} \cdot \vec{B} = 0 \ ,$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{D} \ ,$$

$$\vec{B} = \mu_0 \ (\vec{H} + \vec{M}) \ .$$

Insertion of the material equations into Maxwell's equations leads to

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial}{\partial t} (\vec{H} + \vec{M}) \ ,$$

$$\vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial}{\partial t} (\varepsilon_0 \vec{E} + \vec{P}) \ .$$

Calculation of the scalar products of these vectors with the electric and magnetic field vectors yields

$$\vec{E} \cdot \left(\vec{\nabla} \times \vec{H} \right) = \vec{E} \cdot \vec{j} + \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{P}}{\partial t}$$

$$= \vec{E} \cdot \vec{j} + \frac{\varepsilon_0}{2} \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) + \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} \ ,$$

$$\vec{H} \cdot \left(\vec{\nabla} \times \vec{E} \right) = -\mu_0 \vec{H} \cdot \left(\frac{\partial \vec{H}}{\partial t} + \frac{\partial \vec{M}}{\partial t} \right)$$

$$= -\frac{\mu_0}{2} \frac{\partial}{\partial t} (\vec{H} \cdot \vec{H}) - \mu_0 \vec{H} \cdot \frac{\partial \vec{M}}{\partial t} \ .$$

Using the difference of the two equations,

$$\left[\vec{E} \cdot \left(\vec{\nabla} \times \vec{H} \right) \right] - \left[\vec{H} \cdot \left(\vec{\nabla} \times \vec{E} \right) \right]$$

$$= \vec{E} \cdot \vec{j} + \frac{\partial}{\partial t} \left(\frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{\mu_0}{2} \vec{H} \cdot \vec{H} \right) + \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} + \mu_0 \vec{H} \cdot \frac{\partial \vec{M}}{\partial t}$$

and applying the following vector relation to the left hand side,

$$\vec{c}\ \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a}) - \vec{a} \cdot (\vec{c} \times \vec{b})$$

with $\vec{c} \equiv \vec{\nabla}\ ,\ \vec{a} \equiv \vec{H}\ ,\ \vec{b} \equiv \vec{E}$

$$\Rightarrow \quad \vec{\nabla} \cdot \left(\vec{H} \times \vec{E}\right) = -\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = \vec{E} \cdot \left(\vec{\nabla} \times \vec{H}\right) - \vec{H} \cdot \left(\vec{\nabla} \times \vec{E}\right),$$

we get the result (*)

$$-\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = \vec{E} \cdot \vec{j} + \frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\vec{E} \cdot \vec{E} + \frac{\mu_0}{2}\vec{H} \cdot \vec{H}\right) + \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{M}}{\partial t}\ .$$

The dimension of energy density is energy per volume

$$\left[\left|\vec{\nabla} \cdot \left(E \times \vec{H}\right)\right|\right] = \frac{1}{\mathrm{m}}\frac{\mathrm{V}}{\mathrm{m}}\frac{\mathrm{A}}{\mathrm{m}} = \frac{\mathrm{W}}{\mathrm{m}^3}\ .$$

In the next step we apply the Gauß theorem to the above equation (*)

$$\int_V (\vec{\nabla} \cdot \vec{\alpha})\, dV = \oint_{A(V)} \vec{\alpha} \cdot \hat{n}\ dA\ .$$

Here integration is either over the volume V or the corresponding surface A of V. Application of the Gauß theorem leads to the Pointing theorem

$$-\int_V \vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) dV = -\oint_{A(V)} \left(\vec{E} \times \vec{H}\right) \cdot \hat{n}\ dA \equiv \int_V \text{right hand side of (*)}\ dV$$

The total energy flux into the volume V is

$$-\oint_{A(V)} \left(\vec{E} \times \vec{H}\right) \cdot \hat{n}\ dA = -\oint_{A(V)} \vec{S} \cdot \hat{n}\ dA\ .$$

with Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ (energy flux per unit area). The time average $< \vec{S} >$ is proportional to intensity, where averaging is over times $t >> 2\pi / \omega$

$$I = \frac{1}{2}\,\mathrm{Re}\,(|< \vec{S} >|) = \frac{1}{2}\,\mathrm{Re}\,(|\vec{E} \times \vec{H}^*|)\ .$$

As a result, the electromagnetic field has energy contributions from

$$\int_V \left(\vec{E} \cdot \vec{j}\right) dV \qquad \begin{cases} \text{energy from } \vec{E} \text{ field due to displacement} \\ \text{of free charges (electrons or ions)} \end{cases}$$

$$\int_V \frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}\vec{E} \cdot \vec{E} + \frac{\mu_0}{2}\vec{H} \cdot \vec{H}\right) dV \qquad \begin{cases} \text{material independent: energy of} \\ \text{electromagnetic field in vacuum} \end{cases}$$

$$\int_V \left(\vec{E} \cdot \frac{\partial \vec{P}}{\partial t} \right) dV \qquad \begin{cases} \text{energy from } \vec{E} \text{ field for excitation or} \\ \text{amplification of electric dipoles in} \\ \text{the medium (increase of potential energie),} \\ \text{decribes light-matter interaction} \end{cases}$$

$$\int_V \left(\vec{H} \cdot \frac{\partial \vec{M}}{\partial t} \right) dV \qquad \begin{cases} \text{energy from } \vec{H} \text{ field for excitation or} \\ \text{amplification of magnetic dipoles} \end{cases}$$

1.4 Dipoles in harmonic fields

From the above investigations we have obtained the energy density of the electric field in media. The time averaged part that is used for a change of the materials' polarisation reads

$$< \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} > \; = \; \frac{< \text{energy} >}{\text{volume}} \; .$$

For simplification we assume an isotropic medium with $\vec{E} \| \vec{P}$

$$E(\vec{r}, t) = \mathrm{Re}\left[E_0(\vec{r}) \exp(i\omega t) \right] \; ,$$

$$P(\vec{r}, t) = \mathrm{Re}\left[P_0(\vec{r}) \exp(i\omega t) \right] \; .$$

The relation between E and P is described by the susceptibility χ

$$P = \varepsilon_0 \, \chi \, E$$

$$\Rightarrow \qquad P = \mathrm{Re}\left[\underbrace{\varepsilon_0 \, \chi \, E_0}_{P_0} \; \exp(i\omega t) \right]$$

$$\Rightarrow \qquad \frac{\partial P}{\partial t} = \mathrm{Re}\left[i\omega P_0 \; \exp(i\omega t) \right]$$

The energy density is thus

$$< \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} > \, = \frac{1}{2} \, \mathrm{Re}\left(\vec{E} \cdot (i\omega \vec{P})^* \right)$$

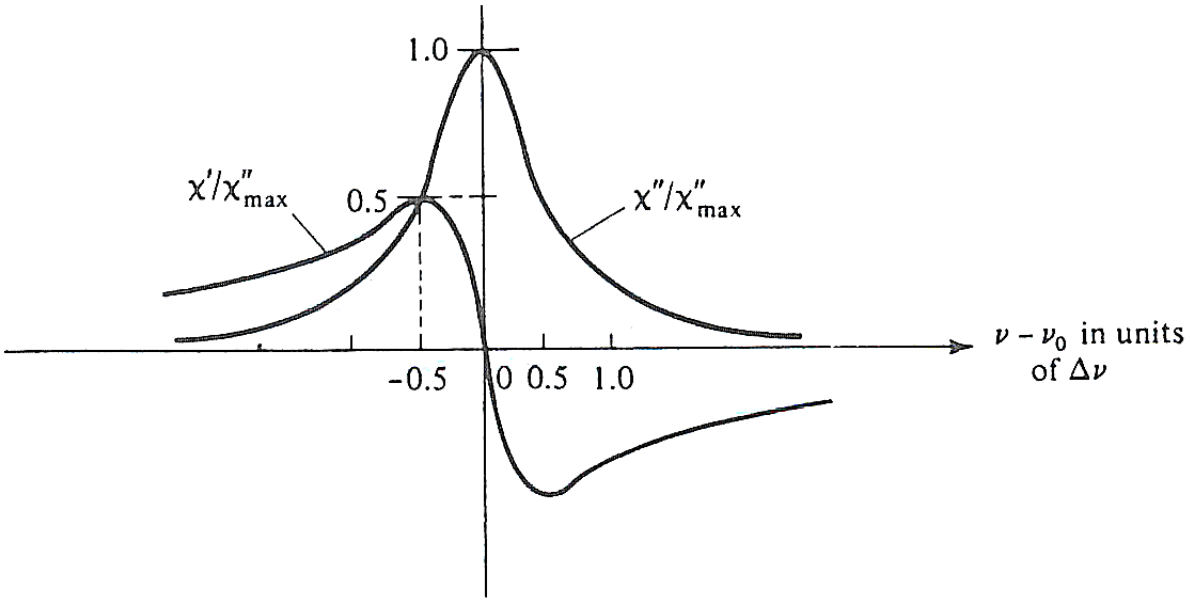
$$= \frac{1}{2} \, \mathrm{Re}\left(\vec{E} \cdot (i\omega \varepsilon_0 \, \chi \, \vec{E})^* \right)$$

$$= \frac{\omega \varepsilon_0}{2} \, \left| \vec{E} \right|^2 \mathrm{Re}(-i\chi^*) \; .$$

In general χ is a complex quantity, i.e. electric field E and polarisation P are not in phase (for χ being a purely imaginary quantity E and P are 90° out of phase)

$$\chi = \chi' - i\chi'' \ .$$

By using $\chi' = n^2 - 1$ and $\chi'' = nc\alpha/\omega$ the real quantities χ' and χ'' are related to refractive index n and absorption coefficient α. The following figure shows the dependence of the susceptibility on frequency (see section 2) in the region of an atomic resonance at frequency ν_0.



As a result we get the necessary energy (density) to induce a change of polarisation of the medium

$$< \vec{E} \cdot \frac{\partial \vec{P}}{\partial t} > = \frac{\omega}{2} \varepsilon_0 \left| \vec{E} \right|^2 \chi''$$

A change of polarisation thus requires $\chi'' \neq 0$: the energy absorbed by the medium is used (in non-magnetic media with μ = 1) for (i) Displacement of free charge carriers and excitation of drift currents $\vec{j} = \sigma \vec{E}$ with conductivity σ (the related energy density is $\int (\vec{E} \cdot \vec{j}) dV$) and (ii) for the excitation of electric dipoles (which finally results in thermal energy or heating of the sample) with the related energy density $1/2\omega\varepsilon_0 \left|\vec{E}\right|^2 \chi''$.

The intensity of the wave can be calculated using the Poynting vector

$$I = < \vec{S} > = \frac{1}{2} \mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)$$

$$= \frac{1}{2} \mathrm{Re}\left(\vec{E} \cdot \left(\frac{1}{\eta} \vec{E} \right)\right) \quad = \frac{1}{2\,\eta} \left| \vec{E} \right|^2 , \qquad \text{(for TEM waves)}$$

with $$\eta = \sqrt{\frac{\mu_0}{\varepsilon\,\varepsilon_0}} = \frac{1}{\sqrt{\varepsilon}}\;\eta_0\;,$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega\;.$$ (vacuum impedance)

With the refractive index $\sqrt{\varepsilon} = n$ of the medium we get

$$I = \frac{n}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\left|\vec{E}\right|^2 = \frac{c\,n\,\varepsilon_0}{2}\left|\vec{E}\right|^2\;,$$

with speed of light in vacuum c

$$c = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}}\;.$$

An alternative description using the magnetic field $\vec{H}$ is

$$I = \frac{1}{2\,c\,n\,\varepsilon_0}\left|\vec{H}\right|^2\;.$$

1.5 Wave propagation in anisotropic media

In general the physical properties of solid matter are direction dependent or anisotropic. Crystals consist of regularly spaced atoms or ions that are arranged in lattice planes. This leads to an anisotropic distribution of the electronic density in the material, which finally results in anisotropic material properties.

In anisotropic media the relations among physical properties are expressed by tensors. A well known example is the relation of polarisation and electric field, which can be expressed by the second-rank tensor of the susceptibility

$$\vec{P} = \varepsilon_0\,\vec{\chi}\,\vec{E}$$

$$P_x = \varepsilon_0\;(\chi_{11}\,E_x + \chi_{12}\,E_y + \chi_{13}\,E_z)$$
$$P_y = \varepsilon_0\;(\chi_{21}\,E_x + \chi_{22}\,E_y + \chi_{23}\,E_z)$$
$$P_z = \varepsilon_0\;(\chi_{31}\,E_x + \chi_{32}\,E_y + \chi_{33}\,E_z)$$

or in short $$P_i = \varepsilon_0\,\chi_{ij}\,E_j$$

Here we have used the sum convention, i.e. summarizing over those indices that appear twice (or multiples of twice). The tensor χ can be transformed to its diagonal form, i.e. there exists a coordinate system (x', y', z') in which the non-diagonal components $(\chi_{ij}\,,\, i \neq j)$ of the tensor disappear:

$$P_{x'} = \varepsilon_0\,\chi_{11}\,E_{x'}\;,\;\; P_{y'} = \varepsilon_0\,\chi_{22}\,E_{y'}\;,\;\; P_{z'} = \varepsilon_0\,\chi_{33}\,E_{z'}\;.$$

The corresponding directions (x', y', z') are called principal axes of the crystal. In what follows we will use this principal coordinate system. Another tensor relation is the dependence of dielectric displacement $\vec{D}$ on electric field $\vec{E}$

$$\vec{D} = \varepsilon_0 \, \hat{\varepsilon} \, \vec{E}$$

with the dielectric tensor $\hat{\varepsilon}$:

$$D_{x'} = \varepsilon_0 \, \varepsilon_{11} \, E_{x'} \quad , \quad D_{y'} = \varepsilon_0 \, \varepsilon_{22} \, E_{y'} \quad , \quad D_{z'} = \varepsilon_0 \, \varepsilon_{33} \, E_{z'} \ .$$

With $\vec{D} = \varepsilon_0 \, \vec{E} + \vec{P}$ follows

$$\varepsilon_{11} = (1 + \chi_{11}) \ , \ \varepsilon_{22} = (1 + \chi_{22}) \ , \ \varepsilon_{33} = (1 + \chi_{33}) \ .$$

In the principal system an electric field E_x of the light wave along the x direction induces a polarisation P_x

$$P_x = \varepsilon_0 \, \chi_{11} \, E_x = \varepsilon_0 \, (\varepsilon_{11} - 1) \, E_x \ .$$

The corresponding phase velocity in the medium is

$$c = \frac{1}{\sqrt{\mu_0 \, \varepsilon_0 \, \varepsilon}} \equiv \frac{1}{\sqrt{\mu_0 \, \varepsilon_0 \, \varepsilon_{11}}} \ .$$

Consequently the wave will „see“ a refractive index $n_1 \equiv \sqrt{\varepsilon_{11}}$. In anisotropic crystals the elements $\varepsilon_{11}, \varepsilon_{22}$ und ε_{33} may be different. Here birefringence is a direct consequence of the anisotropic dielectric properties. A well known example is calcite with $\varepsilon_{11} = \varepsilon_{22} = 2.75, \ \varepsilon_{33} = 2.21$ at a wavelength of 589 nm, which is the sodium D line.

In the principal system the diagonal elements of the dielectric tensor can be related to the corresponding refractive indices. For an optically uniaxial crystal this is

$$n_o = \sqrt{\varepsilon_{11}} = \sqrt{\varepsilon_{22}} \qquad \text{: ordinary refractive index}$$

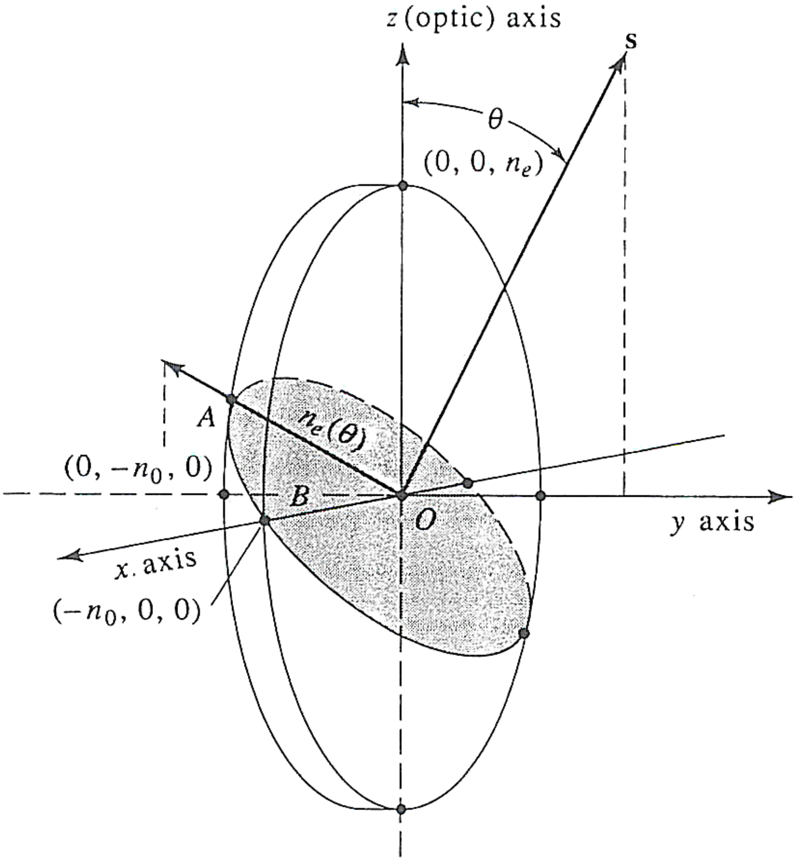
$$n_e = \sqrt{\varepsilon_{33}} \qquad \text{: extraordinary refractive index}$$

A material is called negative birefringent if $n_e < n_o$.

The use of an index ellipsoid is a helpful tool to determine n and ε for a wave with arbitrary propagation direction $\vec{s}$. The corresponding equation of an index ellipsoid becomes

$$\frac{x^2}{\varepsilon_{11}} + \frac{y^2}{\varepsilon_{22}} + \frac{z^2}{\varepsilon_{33}} = 1 \ = \ \frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1 \ .$$

The major axis of this ellipsoid is $2c = 2n_e$, whereas that of the transverse x and y axes is $2a = 2b = 2n_o$.



The two axes of the intersection ellipse (shaded plane) perpendicular to the propagation direction $\vec{s}$ are the refractive indices of the ordinarily and extraordinarily polarized waves n_o and $n_e(\theta)$. It follows that n_o does not depend on the angle θ if the polarization is perpendicular to the z *axis* (the optical axis). On the other hand $n_e = n_e(\theta)$ depends on the propagation direction by

$$n_e^2(\theta) = y^2 + z^2 \ ,$$

$$\sin\theta = z / n_e(\theta) \ .$$

The equation of the index ellipsoid in the yz plane is

$$\frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1$$

$$\frac{1}{n_e^2(\theta)} = \frac{\cos^2(\theta)}{n_o^2} + \frac{\sin^2(\theta)}{n_e^2}$$

The magnitude of the birefringence $n_o - n_e(\theta)$ decreases from a maximum value $n_o - n_e$ for θ = 90° (propagation perpendicular to the optical axis) to zero for θ = 0° (degenerate case, propagation along the optical axis). This is important for the phase matching condition necessary in optical frequency mixing.